Table III : Laplace Transforms

S. No.	Laplace Transform i.e. L	Inverse Laplace Transform i.e. L^{-1}	Remarks
1.	$L f(s) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$	$L^{-1} [\bar{f}(s)] = f(t)$	Definition
2.	$a[f(t)] + b[g(t)] - ch(t)]$	$a\bar{f}(s) + b\bar{g}(s) - c\bar{h}(s)$	Linearity property
3.	$e^{at} f(t)$	$\bar{f}(s-a)$	First shifting property
4.	$f(at)$	$\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$	Change of scale
5.	$f'(t)$	$s\bar{f}(s) - f(o)$	Derivative
6.	$f''(t)$	$s^n \bar{f}(s) - s^{n-1} f'(o) \ldots\ldots f^{n-1}(o)$	nth derivative
7.	$\int_0^t f(u)\, du$	$\frac{1}{s}\bar{f}(s)$	Integral division by s
8.	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\bar{f}$, $n = 1, 2, 3, \ldots\ldots$	Multiplication by t^n
9.	$\frac{1}{t} f(t)$	$\int_0^\infty \bar{f}(s)\, ds$	Division by t
10.	$\int_0^t f(u)\, g(t-u)\, du = F^* G$	$L^{-1}[\bar{f}(s)\, \bar{g}(s)]$	Convolution theorem
11.	1	$1/s$	$s > 0$
12.	t^n	$n!/s^{n+1}$, $n = 1, 2, 3$	Otherwise : $\frac{\Gamma(n+1)}{s^{n+1}}$
13.	e^{at}	$1/(s-a)$	$s > a$
14.	$e^{at} t^n$	$n!/(s-a)^{n+1}$	$s > a$
15.	$\sin at$	$a/(s^2 + a^2)$	$s > 0$
16.	$\cos at$	$s/(s^2 + a^2)$	$s > 0$
17.	$\sinh at$	$a/(s^2 - a^2)$	$s > \vert a \vert$
18.	$\cosh at$	$s/(s^2 - a^2)$	$s > \vert a \vert$
19.	$e^{at} \sin bt$	$b/[(s-a)^2 + b^2]$	$s > a$
20.	$e^{at} \cos bt$	$(s-a)/[(s-a)^2 + b^2]$	$s > a$
21.	$e^{at} \sinh bt$	$b/[(s-a)^2 - b^2]$	$s > a$
22.	$e^{at} \cosh bt$	$(s-a)/[(s-a)^2 - b^2]$	$s > a$
23.	$\frac{1}{2a}(t \sin at)$	$s/(s^2 + a^2)^2$	
24.	$(t \cos at)$	$(s^2 - a^2)/(s^2 + a^2)^2$	
25.	$\frac{1}{2a^3}(\sin at - at \cos at)$	$1/(s^2 + a^2)^2$	
26.	$\frac{1}{2a}(\sin at + at \cos at)$	$s^2/(s^2 + a^2)^2$	
27.	$\cos at - \frac{1}{2} at \sin at$	$s^3/(s^2 + a^2)^2$	
28.	$\frac{1}{2a} t \sinh at$	$s/(s^2 - a^2)^2$	
29.	$t \cosh at$	$(s^2 + a^2)/(s^2 - a^2)^2$	
30.	$\frac{1}{2a}(\sinh at + at \cosh at)$	$s^2/(s^2 - a^2)^2$	

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① Shifting Property : If $L\{f(t)\} = \bar{f}(s)$ then,

$$\mathcal{L}\{e^{at} \cdot f(t)\} = \bar{f}(s-a)$$

② Division Property : If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then,

$$\mathcal{L}\left\{\frac{1}{t} \cdot f(t)\right\} = \int_s^\infty \bar{f}(s) \cdot ds$$

③ Transform of integral property :
If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathcal{L}\left\{\int_0^t f(t) \cdot dt\right\} = \frac{1}{s} \cdot \bar{f}(s)$$

④ Multiplication Property : If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n}[\bar{f}(s)]$$

⑤ Laplace transform of a derivative :
If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathcal{L}\{f^n \cdot (t)\} = s^n \cdot \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} \cdot f''(0) \cdots$$

change of scale

$$\mathcal{L}\{f(at)\} = \frac{1}{a} \cdot \bar{f}\left(\frac{s}{a}\right)$$

Inverse Laplace Transform

FORMULAE:

1) $\mathcal{L}^{-1}\{1/s\} = 1$

2) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3) $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

4) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

5) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\cdot \sin at$

6) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

7) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\cdot \sinh at$

8)

Properties of I.L.T.

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$

1. $\mathcal{L}^{-1}\{\bar{f}(s-a)\} = e^{at}\cdot f(t) \rightarrow$ Shifting property

2. $\mathcal{L}^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{1}{t}\cdot f(t) \rightarrow$ Integral property

3. $\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt \rightarrow$ Division property

4. $\mathcal{L}^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t\cdot f(t) \rightarrow$ Derivative property

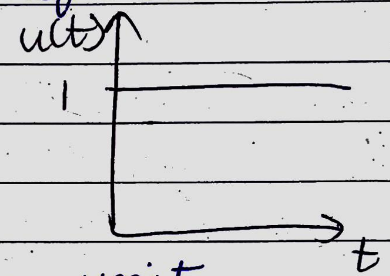
Convolution Theorem

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ and $\mathcal{L}^{-1}\{\bar{g}(s)\} = g(t)$,
then $\mathcal{L}^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\cdot du$.

Unit Step Function (Heaviside's func^n)

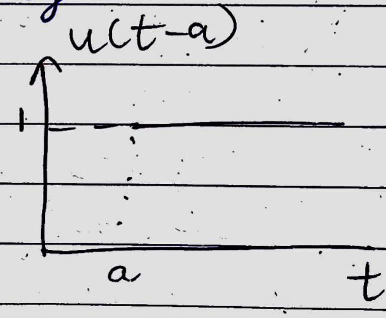
Unit step function is defined as

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$$



The delayed or displaced unit step function $u(t-a)$ represents the func^n $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases}$$



Laplace Transform of Unit Step Function

1. LT of unit step func^n $u(t)$:

$\because$ We know, $u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$

$\therefore \mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}\cdot u(t)\,dt$

$$= \int_0^\infty e^{-st}\cdot dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

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② LT of displaced unit step function $u(t-a)$

$\because$ We know, $u(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases}$

$\therefore \mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} \cdot u(t-a) \cdot dt$

$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$

$= \left| \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s}$

③ LT of function $f(t) \cdot u(t-a)$

$f(t) \cdot u(t-a) = \begin{cases} 0, & t<a \\ f(t), & t \geq a \end{cases}$

then $\mathcal{L}\{f(t) \cdot u(t-a)\} = e^{-as} \, \mathcal{L}\{f(t+a)\}$

④ LT of function, $f(t-a) \, u(t-a)$

$\because f(t-a) \cdot u(t-a) = \begin{cases} 0, & t<a \\ f(t-a), & t \geq a \end{cases}$

then, $\mathcal{L}\{f(t-a) \cdot u(t-a)\}$

$= e^{-as} \cdot \bar{f}(s)$

Laplace transform of periodic functions

A function $f(t)$ is said to be periodic if there exists a constant T $(T>0)$ such that $f(t+T)=f(t)$, for all values of t.

$f(t+2T)=f(t)$

In general, $f(t+n \cdot T)=f(t)$ for all t, where n is an integer and T is the period of the function.

If $f(t)$ is a periodic function with period T i.e. $f(t+T)=f(t)$, then

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st} \cdot f(t) \cdot dt$$